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NOTE

A NOTE ON VARIATIONAL PRINCIPLES IN ELASTICITY

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WE ARE concerned in what follows with a class of extensions of known variational principles for boundary value problems of the differential equations of elasticity

$$\sigma_{x,x}+\tau_{xy,y}+\tau_{xz,z} = 0, \text{ etc.} \tag{1}$$

together with

$$\sigma_x = \partial A_d/\partial \varepsilon_x, \qquad \tau_{xy} = \partial A_d/\partial \gamma_{xy}, \text{ etc.} \tag{2a}$$

or

$$\varepsilon_x = \partial A_s/\partial \sigma_x, \qquad \gamma_{xy} = \partial A_s/\partial \tau_{xy}, \text{ etc.} \tag{2b}$$

In this A_d and A_s are given functions of the six variables ε_x, γ_{xy}, etc. or σ_x, τ_{xy}, etc., respectively, these variables being defined in such a way that

$$\varepsilon_x = u_{x,x}, \qquad \gamma_{xy} = u_{x,y}+u_{y,x}, \text{ etc.} \tag{3}$$

in terms of displacement components, and

$$\tau_{xy} = \tau_{yx}, \text{ etc.} \tag{4}$$

Omitting for the sake of brevity a discussion of boundary conditions we have that equations (1) and (2b) are the Euler equations of a variational equation of the form $\delta \int F_1 dV = 0$ where

$$F_1 = \sigma_x\varepsilon_x+\tau_{xy}\gamma_{xy}+\cdots-A_s \tag{5}$$

with $\varepsilon_x, \gamma_{xy}, \ldots, \tau_{xy}, \tau_{yx}, \ldots$ defined in accordance with (3) and (4) and with the six components of stress and the three components of displacement being varied independently. Imposition of appropriate partial constraint on the nine variation variables reduces the general variational equation involving F_1 to the minimum principles of complementary and potential energy, respectively [1].

A more general variational equation than the one for stresses and displacements has been formulated by Washizu [2], by changing the status of the defining equations for strains (3) into additional Euler equations of the variational equation. Washizu's equation is of the form $\delta \int F_2 \, dV = 0$ where

$$F_2 = \sigma_x(\varepsilon_x-u_{x,x})+\tau_{xy}(\gamma_{xy}-u_{x,y}-u_{y,x})+\cdots-A_d \tag{6}$$

and where now the six components of strain, the six components of stress and the three components of displacement are varied independently.

The principal purpose of the present note is the formulation of two variational equations in which the three moment equilibrium equations (4) for components of stress are Euler equations of a variational principle, rather than equations of definition. This purpose is accomplished by defining nine components of strain in terms of three components of linear displacement and three additional components of angular displacement, as follows

$$\varepsilon_x = u_{x,x}, \qquad \varepsilon_{xy} = u_{x,y}+\omega_z, \qquad \varepsilon_{yx} = u_{y,x}-\omega_z, \text{ etc.} \tag{7}$$

and by replacing the six stress–strain relations (2) by nine stress–strain relations of the form

$$\sigma_x = \partial B_d/\partial \varepsilon_x, \qquad \tau_{xy} = \partial B_d/\partial \varepsilon_{xy}, \text{ etc.} \tag{8a}$$

or

$$\varepsilon_x = \partial B_s/\partial \sigma_x, \qquad \varepsilon_{xy} = \partial B_s/\partial \tau_{xy}, \text{ etc.} \tag{8b}$$

It is now readily seen that the variational equation involving F_1 may be generalized to a variational equation $\delta \int G_1 \, dV = 0$ where

$$G_1 = \sigma_x\varepsilon_x+\tau_{xy}\varepsilon_{xy}+\tau_{yx}\varepsilon_{yx}+\cdots-B_s \tag{9}$$

with ε_x, ε_{xy}, etc., defined by (7) and with the nine components of stress σ_x, τ_{xy}, τ_{yx}, etc., and the six components of displacement u_x, ω_x, etc., being varied independently.

Analogously, the variational equation involving F_2 may be generalized to a variational equation $\delta\int G_2\,\mathrm{d}V = 0$ where

$$G_2 = \sigma_x(\varepsilon_x - u_{x,x}) + \tau_{xy}(\varepsilon_{xy} - u_{x,y} + \omega_z) + \tau_{yx}(\varepsilon_{yx} - u_{y,x} - \omega_z) + \cdots - B_d \tag{10}$$

and where now there are altogether nine independent strain variations, nine independent stress variations and six independent displacement variations.

Further generalizations of the variational equations involving G_1 and G_2, so as to take account of body forces, finite deflections, time dependence, couple stresses, etc. may be obtained by following the same procedures as with F_1 and F_2.

It remains to establish the form of the functions $B_d(\varepsilon_x, \varepsilon_{xy}, \varepsilon_{yx}, \ldots)$ and $B_s(\sigma_x, \tau_{xy}, \tau_{yx}, \ldots)$ which corresponds to the form of the functions $A_d(\varepsilon_x, \gamma_{xy}, \ldots)$ and $A_s(\sigma_x, \tau_{xy}, \ldots)$ for the case that the moment equilibrium equations (4) are equations of definition rather than Euler equations.

In order to determine B_d we take account of the fact that the stress–strain relations (8a) should be compatible with the Euler moment equilibrium equations. Accordingly we have the relations

$$\partial B_d/\partial\varepsilon_{xy} - \partial B_d/\partial\varepsilon_{yx} = 0, \text{ etc.} \tag{11a}$$

and these, as partial differential equations in the independent variables ε_{xy}, ε_{yx}, etc., imply that B_d depends on the variables ε_{xy}, ε_{yx}, etc., as a function of the sums $\varepsilon_{xy}+\varepsilon_{yx}$, etc. Since $\varepsilon_{xy}+\varepsilon_{yx} = \gamma_{xy}$ we may then identify the function B_d with the customary function A_d, except that in this γ_{xy} needs to be considered explicitly as the combination $\varepsilon_{xy}+\varepsilon_{yx}$, with $\delta\varepsilon_{xy}$ and $\delta\varepsilon_{yx}$ as independent variations.

In order to determine the form of B_s we take account of the fact that the angular displacement quantities ω_z, etc., and the linear displacement quantities u_x, etc., should satisfy the relations

$$\omega_z = \tfrac{1}{2}(u_{y,x} - u_{x,y}), \text{ etc.} \tag{12}$$

In view of equation (7) this means that the stress–strain relations (8b) should be compatible with the relations $\varepsilon_{yx} - \varepsilon_{xy} = 0$, etc., or with

$$\partial B_s/\partial\tau_{xy} - \partial B_s/\partial\tau_{yx} = 0, \text{ etc.} \tag{11b}$$

The partial differential equations (11b) indicate that B_s must depend on the six independent variables τ_{xy}, τ_{yx}, etc. as a function of the sums $\tau_{xy}+\tau_{yx}$, etc. Accordingly we have that B_s is obtained from A_s by replacing τ_{xy}, τ_{xz}, τ_{yz} in it by $\frac{1}{2}(\tau_{xy}+\tau_{yx})$, $\frac{1}{2}(\tau_{xz}+\tau_{zx})$, $\frac{1}{2}(\tau_{yz}+\tau_{zy})$.

In order to illustrate the meaning of this requirement we consider the case of a linear isotropic medium. For this case τ_{xy}, etc. occurs in A_s in the combination

$$\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2 - \sigma_x\sigma_y - \sigma_x\sigma_z - \sigma_y\sigma_z.$$

A corresponding invariant term in B_s is

$$\tau_{xy}\tau_{yx} + \tau_{xz}\tau_{zx} + \tau_{yz}\tau_{zy} - \sigma_x\sigma_y - \sigma_x\sigma_z - \sigma_y\sigma_z + k[(\tau_{xy}-\tau_{yx})^2 + (\tau_{xz}-\tau_{zx})^2 + (\tau_{yz}-\tau_{zy})^2]$$

with k an arbitrary constant.

The requirement that B_s be a function of the sums $\tau_{xy}+\tau_{yx}$, etc. means that, necessarily, we must have

$$k = \tfrac{1}{4}.$$

Possible applications of the variational equations $\sigma\int G_1\,\mathrm{d}V = 0$ or $\delta\int G_2\,\mathrm{d}V = 0$ depend on the idea that it may be of advantage, in the approximate solution of problems by the direct methods of the calculus of variations, to satisfy moment as well as force equilibrium equations approximately only, instead of satisfying one set exactly and the other approximately. One such application of a special form of the equation $\delta\int G_1\,\mathrm{d}V = 0$ is to the derivation of approximate stress–strain relations for thin elastic shells [3]. Further considerations may be found in [4].

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Zusammenfassung—Bekannte Variationsprinzipe der Elastizitätstheorie für (1) Spannungen und Verschiebungen, (2) Spannungen, Formänderungen und Verschiebungen werden verallgemeinert in solcher Weise, dass die Momentengleichgewichtsbedingungen $\tau_{xy} = \tau_{yx}$, etc., als Euler'sche Gleichungen der Variationsprinzipe erscheinen, und nicht als Definitionsgleichungen. Anwendungen dieser Ergebnisse beruhen auf der Idee dass es manchmal von Nutzen sein kann, bei der Aufstellung von Näherungslösungen von Randwertproblemen, Moment und Kraftgleichgewichtsbedingungen beide nur näherungsweise zu befriedigen.

Абстракт—Известные принципы теории упругости для: (1) напряжений и смещений и (2) напряжений, деформаций и смещений обобщаются так, чтобы получить условия для равновесия моментов, $\tau_{xy} = \tau_{yx}$ и т.д., в форме Эйлерова уравнения для вариационных принципов скорее, чем в форме определенных уравнений. Применение этих результатов основано на идее, что при составлении приблизительных решений для проблем граничных значений, иногда важнее удовлетворить только приблизительно условия равновесия и для моментов и для сил.